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Descents, inversions, and major indices in permutation groups

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Abstract

A multivariate generating function involving the descent, major index, and inversion statistic first given by Ira Gessel is generalized to other permutation groups. We provide generating functions for variants of these three statistics for the Weyl groups of type B and D , wreath product groups, and multiples of permutations. All of our ideas are combinatorial in nature and exploit fundamental relationships between the elementary and homogeneous symmetric functions.

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1. Introduction and preliminaries

A permutation statistic is a function mapping permutations to nonnegative integers. The modern analysis of such objects began in the early twentieth century with the work of MacMahon [11]. He popularized the “classic” notions of the descent, inversion, and major index statistics. They are defined such that if $\sigma = \sigma_1 \cdots \sigma_n$ is an element of the symmetric group S_n written in one line notation, then

$$\text{des}(\sigma) = \sum_{i=1}^{n-1} \chi(\sigma_i > \sigma_{i+1}), \quad \text{inv}(\sigma) = \sum_{i < j} \chi(\sigma_i > \sigma_j) \quad \text{and} \quad \text{maj}(\sigma) = \sum_{i=1}^{n-1} i \chi(\sigma_i > \sigma_{i+1}),$$

where for any statement A , $\chi(A)$ is 1 if A is true and 0 if A is false. In addition to those elements in the symmetric group, these definitions hold for any finite sequence of numbers.

Properties of these statistics—together with many subsequent generalizations for various permutation groups—remain an active area of research today. In the past few decades, beautiful combinatorial and bijective proofs of classical and new results have been published.

An abundance of mathematical research has been devoted to the relationship between generating functions and permutation statistics. Most pertinent to our work, Ira Gessel gave a generating function for

$$\sum_{\sigma \in S_n} x^{\text{des}(\sigma)} u^{\text{maj}(\sigma)} q^{\text{inv}(\sigma)} \tag{1}$$

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both in his thesis and in a paper co-authored with Garsia [9,8]. In this paper we will begin by reproving this result. Then, we will show how similar proofs indicate a systematic approach to finding more generating functions for similar permutation statistics on other collections of objects.

We will manipulate basic relationships between the elementary and homogeneous symmetric functions to prove our theorems. The idea of extracting information about permutation statistics through symmetric function theory has been used for decades, but the methods of this paper—defining a homomorphism on the elementary symmetric functions and evaluating it on the homogeneous symmetric functions—was first given by Brenti [4,5]. Beck and the second author reproved his results combinatorially [1–3]. It is this approach which is closest to our own.

The organization of this paper is as follows. Section 2 contains the result of Gessel proved in our new manner. In Section 3, the main proof in Section 2 is adapted to produce information about statistics on the hyperoctahedral group, its subgroup D_n , wreath product groups, and multiples of permutations. Finally, in Section 4, we define a new class of symmetric functions in order to apply our methods and end with a generating function further refining those in Section 3.

We now conclude this section with a short establishment of the definitions needed for our later developments.

A partition $\lambda = (\lambda_1, \dots, \lambda_\ell)$ is a finite sequence of increasing nonnegative integers. Let $\ell(\lambda)$ be the number of nonzero integers in λ . If the sum of these integers is equal to n , then we say λ is a partition of n and write $\lambda \vdash n$. A partition λ is identified with its (French) Ferrers diagram which is $\ell(\lambda)$ rows of left justified squares where the i th row has length λ_i reading top to bottom.

A symmetric polynomial q in the variables x_1, \dots, x_N with coefficients in \mathbb{Q} is a polynomial with the property that

$$q(x_1, \dots, x_N) = q(x_{\sigma(1)}, \dots, x_{\sigma(N)})$$

for all $\sigma \in S_N$. Let \mathcal{A}^N be the set of symmetric polynomials in x_1, \dots, x_N and \mathcal{A}_n^N be the subset of \mathcal{A}^N containing the homogeneous elements of degree n .

Using the ring homomorphism from \mathcal{A}^{N+1} to \mathcal{A}^N defined by taking $x_{N+1} = 0$, let $\mathcal{A}_n = \varprojlim_N \mathcal{A}_n^N$ for each $n \geq 0$ and define $\mathcal{A} = \bigoplus_{n \geq 0} \mathcal{A}_n$. This set \mathcal{A} is known as the ring of symmetric functions and \mathcal{A}_n the set of homogeneous symmetric functions of degree n . By the definition of projective limits, an element in \mathcal{A}_n is a sequence $\{q_n\}_{n \geq 0}$ where $q_n \in \mathcal{A}_n^N$ for each n . This sequence has the property that q_n is found by taking $x_{N+1} = 0$ in q_{n+1} , meaning that we may think of symmetric functions as an infinite series of monomials in an infinite number of variables.

Our technical definition of the ring of symmetric functions is needed to ensure the validity of taking an infinite series of monomials in an infinite number of variables, but symmetric functions may be thought of in a much simpler manner. In short, a symmetric function in the variables x_1, x_2, \dots may be thought of as an infinite sum of monomials with coefficients in \mathbb{Q} with the property that any rearrangement of the variables leaves the symmetric function unchanged.

For example, the n th elementary symmetric function e_n may be defined by

$$\sum_{n \geq 0} e_n t^n = \prod_i (1 + x_i t).$$

For any partition $\lambda = (\lambda_1, \dots, \lambda_\ell)$, let $e_\lambda = e_{\lambda_1} \cdots e_{\lambda_\ell}$. It is well known that $\{e_\lambda : \lambda \text{ is a partition}\}$ is a basis for \mathcal{A} .

The n th homogeneous symmetric function h_n is defined such that

$$\sum_{n \geq 0} h_n t^n = \prod_i \frac{1}{1 - x_i t}.$$

These definitions of the homogeneous and elementary symmetric functions give

$$\sum_{n \geq 0} h_n t^n = \prod_i \frac{1}{1 - x_i t} = \left(\prod_i (1 - x_i t) \right)^{-1} = \left(\sum_{n \geq 0} e_n (-t)^n \right)^{-1}. \quad (2)$$

This trivial restatement of definitions is the key to finding our generating functions.

The expansion of the homogeneous symmetric functions in terms of the elementary symmetric function basis has been described combinatorially by Egecioglu and the second author [7]. A rectangle of height 1 and length n chopped into “bricks” of lengths found in the partition λ is known as a brick tabloid of shape (n) and type λ . For example,



Fig. 1. A brick tabloid of shape (12) and type (2, 3, 7).

Fig. 1 shows one brick tabloid of shape (12) and type (2, 3, 7). Let $B_{\lambda,n}$ be the number of such objects. Through simple recursions, it is not difficult to prove that

$$h_n = \sum_{\lambda \vdash n} (-1)^{n-\ell(\lambda)} B_{\lambda,n} e_{\lambda}. \quad (3)$$

Those unfamiliar with symmetric functions are referred to [10].

Standard notation from hypergeometric function theory will be used throughout this work. For $n \geq 1$ and $\lambda \vdash n$, let

$$[n]_{p,q} = \frac{p^n - q^n}{p - q}, \quad [n]_{p,q}! = [n]_{p,q} \cdots [1]_{p,q} \quad \text{and} \quad \begin{bmatrix} n \\ \lambda \end{bmatrix}_{p,q} = \frac{[n]_{p,q}!}{[\lambda_1]_{p,q}! \cdots [\lambda_\ell]_{p,q}!}$$

be the p, q -analogues of $n, n!$, and $\binom{n}{\lambda}$, respectively. By convention, let $[0]_{p,q} = 0$ and $[0]_{p,q}! = 1$. It is useful to keep in mind that p, q -analogues are symmetric in p and q . In addition, let $(x, y; p, q)_0 = 1$ and

$$(x, y; p, q)_n = (x - y)(xp - yq) \cdots (xp^{n-1} - yq^{n-1}).$$

A p, q -analog for the exponential function is defined by

$$e_{p,q}^t = \sum_{n \geq 0} \frac{t^n}{[n]_{p,q}!} q^{\binom{n}{2}}.$$

2. A generating function for the symmetric group

In this section we will reprove Ira Gessel's generating function for (1) in Theorem 2. In the process, we may as well further refine this generating function by three more statistics. For $\sigma = \sigma_1 \cdots \sigma_n \in S_n$, let

$$\text{ris}(\sigma) = 1 + \sum_{i=1}^{n-1} \chi(\sigma_i < \sigma_{i+1}), \quad \text{coinv}(\sigma) = \sum_{i < j} \chi(\sigma_i < \sigma_j) \quad \text{and} \quad \text{comaj}(\sigma) = n + \sum_{i=1}^{n-1} i \chi(\sigma_i < \sigma_{i+1}).$$

These statistics are referred to as the rise, co-inversion, and co-major index statistics. For $\sigma \in S_n$, it is not difficult to see that $\text{des}(\sigma) + \text{ris}(\sigma) = n$, $\text{inv}(\sigma) + \text{coinv}(\sigma) = \binom{n}{2}$, and $\text{maj}(\sigma) + \text{comaj}(\sigma) = \binom{n+1}{2}$. Thus, these three permutation statistics do not give any new information over the descent, inversion, and major index statistics. We will refine our generating functions with these new statistics, however, because there are results in the literature which prove one theorem for descents, inversions, and major index statistics and a second theorem of the rise, co-inversion, and co-major index statistics. In addition, there are situations where it is convenient to know the distribution of various combinations of these six statistics over the symmetric group and so writing our theorems in the greatest generality is useful.

Lemma 1. For positive integers b_1, \dots, b_ℓ which sum to n ,

$$\begin{bmatrix} n \\ b_1, \dots, b_\ell \end{bmatrix}_{p,q} = \sum_{r \in \mathcal{R}(1^{b_1}, \dots, \ell^{b_\ell})} q^{\text{inv}(r)} p^{\text{coinv}(r)}$$

where $\mathcal{R}(1^{b_1}, \dots, \ell^{b_\ell})$ is the set of rearrangements of b_1 1's, b_2 2's, etc.

Proof. From the definition of $[n]_{p,q}$, $[n]_{p,q} = p^{n-1}[n]_{\frac{q}{p},1}$. Therefore,

$$\left[\begin{matrix} n \\ b_1, \dots, b_\ell \end{matrix} \right]_{q,p} = \frac{p^{(n-1)+\dots+1}}{p^{(b_1-1)+\dots+1} \dots p^{(b_\ell-1)+\dots+1}} \left[\begin{matrix} n \\ b_1, \dots, b_\ell \end{matrix} \right]_{\frac{q}{p},1}.$$

Carlitz showed that $\left[\begin{matrix} n \\ \lambda \end{matrix} \right]_{q,1} = \sum_{r \in \mathcal{R}(1^{b_1}, \dots, \ell^{b_\ell})} q^{\text{inv}(r)}$ where λ is the partition with b_i parts of size i [6]. With this, the above expression may be written to read

$$p^{\binom{n}{2} - \binom{b_1}{2} - \dots - \binom{b_\ell}{2}} \sum_{r \in \mathcal{R}(1^{b_1}, \dots, \ell^{b_\ell})} \frac{q^{\text{inv}(r)}}{p^{\text{inv}(r)}} = \sum_{r \in \mathcal{R}(1^{b_1}, \dots, \ell^{b_\ell})} q^{\text{inv}(r)} p^{\binom{n}{2} - \binom{b_1}{2} - \dots - \binom{b_\ell}{2} - \text{inv}(r)}.$$

The lemma follows by noting that

$$\text{coinv}(r) = \binom{n}{2} - \binom{b_1}{2} - \dots - \binom{b_\ell}{2} - \text{inv}(r). \quad \square$$

Define a ring homomorphism ζ by defining it on the elementary symmetric function e_n such that

$$\zeta(e_n) = \sum_{\substack{i_0, \dots, i_k \geq 0 \\ i_0 + \dots + i_k = n}} \frac{(u/v)^{0i_0 + \dots + ki_k}}{[i_0]_{p,q}! \dots [i_k]_{p,q}!} q^{\binom{i_0}{2} + \dots + \binom{i_k}{2}}.$$

Since products of elementary symmetric functions form a basis for the ring of symmetric functions, the definition of the homomorphism ζ uniquely extends to all other symmetric functions. In particular, understanding the action of ζ on h_n is at the heart of our proof of Theorem 2 below.

Theorem 2.

$$\begin{aligned} & \sum_{n \geq 0} \frac{t^n}{[n]_{p,q}! (y, x; v, u)_{n+1}} \sum_{\sigma \in S_n} x^{\text{des}(\sigma)} y^{\text{ris}(\sigma)} u^{\text{maj}(\sigma)} v^{\text{comaj}(\sigma)} q^{\text{inv}(\sigma)} p^{\text{coinv}(\sigma)} \\ &= \sum_{k \geq 0} \frac{x^k}{y^{k+1} \mathbf{e}_{p,q}^{-t(u/v)^0} \dots \mathbf{e}_{p,q}^{-t(u/v)^k}}. \end{aligned}$$

Proof. First we apply ζ to $[n]_{p,q}! h_n$. We have

$$\begin{aligned} [n]_{p,q}! \zeta(h_n) &= [n]_{p,q}! \sum_{\lambda \vdash n} (-1)^{n-\ell(\lambda)} B_{\lambda,n} \zeta(e_\lambda) \\ &= [n]_{p,q}! \sum_{\lambda \vdash n} (-1)^{n-\ell(\lambda)} B_{\lambda,n} \prod_{j=1}^{\ell(\lambda)} \sum_{\substack{i_{j,0}, \dots, i_{j,k} \geq 0 \\ i_{j,0} + \dots + i_{j,k} = \lambda_j}} \frac{(u/v)^{0i_{j,0} + \dots + ki_{j,k}}}{[i_{j,0}]_{p,q}! \dots [i_{j,k}]_{p,q}!} q^{\binom{i_{j,0}}{2} + \dots + \binom{i_{j,k}}{2}}. \end{aligned} \quad (4)$$

Rewriting p, q -analogues, the last equation is equal to

$$\sum_{\lambda \vdash n} \left[\begin{matrix} n \\ \lambda \end{matrix} \right]_{p,q} (-1)^{n-\ell(\lambda)} B_{\lambda,n} \prod_{j=1}^{\ell(\lambda)} \sum_{\substack{i_{j,0}, \dots, i_{j,k} \geq 0 \\ i_{j,0} + \dots + i_{j,k} = \lambda_j}} \left[\begin{matrix} \lambda_j \\ i_{j,0}, \dots, i_{j,k} \end{matrix} \right]_{p,q} (u/v)^{0i_{j,0} + \dots + ki_{j,k}} q^{\binom{i_{j,0}}{2} + \dots + \binom{i_{j,k}}{2}}. \quad (5)$$

Let us begin to build combinatorial objects from (5) with a brick tabloid T . In each nonterminal cell in each brick, place a -1 . In each terminal cell in a brick, place a 1 . The choice of the partition in which to fill the brick tabloid accounts for the summation and the $B_{\lambda,n}$ term in (5) while the $(-1)^{n-\ell(\lambda)}$ term in (5) is accounted for with the -1 . For each brick in T , choose nonnegative integers i_0, \dots, i_k that sum to the total length of the brick. This accounts for the product and second sum in (5). Using powers of (u/v) , these choices for i_0, \dots, i_k can be recorded in T . In each

-1	-1	1	-1	-1	-1	-1	-1	-1	-1	1	-1	1
u^1v^{-1}	u^1v^{-1}	u^3v^{-3}	u^0v^0	u^0v^0	u^0v^0	u^0v^0	u^2v^{-2}	u^3v^{-3}	u^3v^{-3}	u^1v^{-1}	u^1v^{-1}	

Fig. 2. One possible object when $k = 3$ and $n = 12$.

brick, place a power of (u/v) in each cell such that the powers weakly increase from left to right and the number of occurrences of $(u/v)^j$ is i_j . At this point, we have constructed an object which may look something like Fig. 2.

The only components in (5) which have not been used involve powers of p and q . We will explain how these powers of p and q will fill the cells of T with a permutation of n such that a decrease must occur between consecutive cells labeled with the same power of (u/v) . Along with this permutation of n , powers of p and q will be placed in each cell recording coinversions and inversions.

By Lemma 1, the $[\frac{n}{\lambda}]_{p,q}$ term in (5) gives powers of q and p counting inversions and coinversions of a rearrangement of λ_1 0's, λ_2 1's, etc. We will use this to select which integers in a permutation of n will appear in each brick. Suppose that the size of the bricks read from left to right in T are b_0, \dots, b_k . Consider a rearrangement r of $0^{b_0}, \dots, k^{b_k}$ and construct a permutation σ_r by labeling the 0's from left to right with $1, 2, \dots, b_0$, the 1's from left to right with $b_0 + 1, \dots, b_0 + b_1$, and in general the i 's from left to right with $1 + \sum_{j=1}^{i-1} b_j, \dots, b_i + \sum_{j=1}^{i-1} b_j$. In this way, σ_r^{-1} starts with the positions of the 0's in r in increasing order, followed by the positions of the 1's in r in increasing order, etc. For example, if T has bricks of length 3, 7, 2, one possible rearrangement to consider is $r = 1\ 0\ 1\ 1\ 2\ 0\ 1\ 2\ 1\ 0\ 1\ 1$. Below we display σ_r and σ_r^{-1} .

		1	2	3	4	5	6	7	8	9	10	11	12
r	=	1	0	1	1	2	0	1	2	1	0	1	1
σ_r	=	4	1	5	6	11	2	7	12	8	3	9	10
σ_r^{-1}	=	2	6	10	1	3	4	7	9	11	12	5	8

This tells us that when selecting a permutation of 12 to place in T , the integers 2, 6, 10 should appear in the brick of size 3, the integers 1, 3, 4, 7, 9, 11, 12 should appear in the brick of size 7, and the integers 5, 8 should appear in the brick of size 2. It is not difficult to verify that $\text{inv}(r) = \text{inv}(\sigma_r) = \text{inv}(\sigma_r^{-1})$.

For each brick of length λ_j , there is an unused term of the form $[\frac{\lambda_j}{i_0, \dots, i_k}]_{p,q} q^{(i_0) + \dots + (i_k)}$ (these i_0, \dots, i_k have already been chosen earlier). Lemma 1 enables us to start with a rearrangement a of i_0 0's, i_1 1's, etc. to use the p, q -multinomial coefficient. Record from right to left the 0's in a with $1, \dots, i_0$. Then record the 1's in a from right to left with $i_0 + 1, \dots, i_0 + i_1$. Continue this process k times to form a permutation of λ_j from a , τ_a^{-1} . The inverse, τ_a , records the places of the 0's, 1's, etc., and therefore, must have decreasing sequences of length i_0, \dots, i_k . Let $\overline{\tau}_a$ be the permutation τ_a where the integers $1, \dots, \lambda_j$ have been replaced by whatever integers the factor $[\frac{n}{\lambda}]_{p,q}$ dictates should appear in the j th brick.

For example, if $k = 3$, $i_0 = 4$, $i_1 = 0$, $i_2 = 1$, and $i_3 = 2$, a permutation of 7 may be formed from 0 2 0 3 3 0 0.

Continuing our example from above, the brick of size 7 should contain the integers 1, 3, 4, 7, 9, 11, and 12. The permutations τ_a^{-1} , τ_a , and $\overline{\tau}_a$ can be found:

		1	2	3	4	5	6	7
a	=	0	2	0	3	3	0	0
τ_a^{-1}	=	4	5	3	7	6	2	1
τ_a	=	7	6	3	1	2	5	4
$\overline{\tau}_a$	=	12	11	4	1	3	9	7

By construction, we have that

-1	-1	1	-1	-1	-1	-1	-1	-1	1	-1	1
u^1v^{-1}	u^1v^{-1}	u^3v^{-3}	u^0v^0	u^0v^0	u^0v^0	u^0v^0	u^2v^{-2}	u^3v^{-3}	u^3v^{-3}	u^1v^{-1}	u^1v^{-1}
q^9p^2	q^1p^9	q^4p^5	q^8p^0	q^7p^0	q^2p^4	q^0p^5	q^0p^4	q^3p^0	q^1p^1	q^1p^0	q^0p^0
10	2	6	12	11	4	1	3	9	7	8	5

Fig. 3. An object coming from (5) when $k = 3$ and $n = 12$.

1	-1	1	-1	-1	-1	-1	-1	-1	1	-1	1
u^1v^{-1}	u^1v^{-1}	u^3v^{-3}	u^0v^0	u^0v^0	u^0v^0	u^0v^0	u^2v^{-2}	u^3v^{-3}	u^3v^{-3}	u^1v^{-1}	u^1v^{-1}
q^9p^2	q^1p^9	q^4p^5	q^8p^0	q^7p^0	q^2p^4	q^0p^5	q^0p^4	q^3p^0	q^1p^1	q^1p^0	q^0p^0
10	2	6	12	11	4	1	3	9	7	8	5

Fig. 4. The image of Fig. 3.

1	1	1	1	1	1	1	1	1	1	1	1
u^3v^{-3}	u^3v^{-3}	u^3v^{-3}	u^2v^{-2}	u^2v^{-2}	u^1v^{-1}	u^1v^{-1}	u^1v^{-1}	u^1v^{-1}	u^1v^{-1}	u^0v^0	u^0v^0
q^3p^8	q^4p^6	q^5p^4	q^1p^7	q^1p^6	q^1p^5	q^1p^4	q^1p^3	q^2p^1	q^2p^0	q^0p^1	q^0p^0
4	6	8	2	3	5	7	9	11	12	1	10

Fig. 5. A fixed point when $k = 3$ and $n = 12$.

$$\text{inv}(\bar{\tau}_a) = \text{inv}(\tau_a) = \text{inv}(\tau_a^{-1}) = \text{inv}(a) + \binom{i_0}{2} + \cdots + \binom{i_k}{2}.$$

Therefore, for each brick of size λ_j , we may associate a permutation of λ_j such that the permutation must have a descent if two consecutive cells have the same power of (u/v) . By taking along a power of $q^{\text{inv}(\bar{\tau}_a)}$ and $p^{\text{coinv}(\bar{\tau}_a)}$, we are able to account for the factors in (5) of the form $[i_0, \dots, i_k]_{p,q} q^{\binom{i_0}{2} + \cdots + \binom{i_k}{2}}$. Every term in (5) has now been used.

Fig. 3 gives one example of such an object created in this manner. These decorated brick tabloids of shape (n) and type λ for some $\lambda \vdash n$ have the following properties: the cells in each brick contain -1 except for the final cell which contains 1, each cell contains a power of (u/v) such that the powers weakly increase within each brick, T contains a permutation of n which must have a decrease between consecutive cells within a brick if the cells are marked with the same power of (u/v) , and each cell contains a power of q counting the number of cells to the right which are smaller and a power of p counting the number of cells to the right which are larger.

The weight of such an object is defined to be the product of all (u/v) , p , q , and -1 labels. In this way, $[n]_{p,q}!^{\xi}(h_n)$ is the weighted sum over all possible decorated brick tabloids. A sign-reversing weight-preserving involution will rid ourselves of any T with a negative weight.

Scan the cells from left to right looking for either a cell containing -1 or two consecutive bricks which may be combined to preserve the properties of this collection of objects. If a -1 is scanned first, break the brick containing the -1 into two immediately after the violation and change the -1 to 1. If the second situation is scanned first, glue the bricks together and change the 1 in the middle to -1 . For example, the image of Fig. 3 is displayed in Fig. 4.

By definition, this is a sign-reversing weight-preserving involution with fixed points such that: there are no bricks containing -1 and therefore every brick must be of length 1, the powers of (u/v) weakly decrease, and if two consecutive bricks have the same power (u/v) , then the permutation must increase there. One example of a fixed point may be found in Fig. 5. We now turn our attention to counting fixed points.

Suppose that the powers of (u/v) in a fixed point are r_1, \dots, r_n when read from left to right. It must be the case that $k \geq r_1 \geq \cdots \geq r_n$. Define nonnegative integers a_i by $a_i = r_i - r_{i+1}$ for $i = 1, \dots, n-1$ and let $a_n = r_n$. It follows that $r_1 + \cdots + r_n = a_1 + 2a_2 + \cdots + na_n$, $a_1 + \cdots + a_n = r_1 \leq k$, and if σ is the permutation in a fixed point, $a_i \geq \chi(\sigma_i > \sigma_{i+1})$.

In this way, the weighted sum over all fixed points—and therefore $[n]_{p,q}!\zeta(h_n)$ —is equal to

$$\begin{aligned} & \sum_{\sigma \in S_n} q^{\text{inv}(\sigma)} p^{\text{coinv}(\sigma)} \sum_{\substack{a_1 + \dots + a_n \leq k \\ a_i \geq \chi(\sigma_i > \sigma_{i+1})}} (u/v)^{a_1 + 2a_2 + \dots + na_n} \\ &= \sum_{\sigma \in S_n} q^{\text{inv}(\sigma)} p^{\text{coinv}(\sigma)} \sum_{a_1 \geq \chi(\sigma_1 > \sigma_2)} \dots \sum_{a_n \geq \chi(\sigma_n > n+1)} (x/y)^{a_1 + \dots + a_n} (u/v)^{a_1 + 2a_2 + \dots + na_n} \Bigg|_{(x/y) \leq k}, \end{aligned}$$

where $\text{expression}|_{t \leq k}$ means to sum the coefficients of t^j for $j = 0, \dots, k$ in expression . Rewriting the above equation, we have

$$\begin{aligned} & \sum_{\sigma \in S_n} q^{\text{inv}(\sigma)} p^{\text{coinv}(\sigma)} \sum_{a_1 \geq \chi(\sigma_1 > \sigma_2)} ((x/y)(u/v))^{a_1} \dots \sum_{a_n \geq \chi(\sigma_n > n+1)} ((x/y)(u/v)^n)^{a_n} \Bigg|_{(x/y) \leq k} \\ &= \sum_{\sigma \in S_n} q^{\text{inv}(\sigma)} p^{\text{coinv}(\sigma)} \frac{((x/y)(u/v))^{\chi(\sigma_1 > \sigma_2)} \dots ((x/y)(u/v)^n)^{\chi(\sigma_n > n+1)}}{(1 - (x/y)(u/v)) \dots (1 - (x/y)(u/v)^n)} \Bigg|_{(x/y) \leq k} \\ &= \sum_{\sigma \in S_n} q^{\text{inv}(\sigma)} p^{\text{coinv}(\sigma)} \frac{(x/y)^{\text{des}(\sigma)} (u/v)^{\text{maj}(\sigma)}}{(1 - (x/y)(u/v)) \dots (1 - (x/y)(u/v)^n)} \Bigg|_{(x/y) \leq k}. \end{aligned}$$

Dividing by $(1 - x/y)$ and factoring the powers of y and v out of the denominator allows the above expression to be rewritten as

$$\sum_{\sigma \in S_n} q^{\text{inv}(\sigma)} p^{\text{coinv}(\sigma)} \frac{x^{\text{des}(\sigma)} y^{n - \text{des}(\sigma) + 1} u^{\text{maj}(\sigma)} v^{\binom{n+1}{2} - \text{maj}(\sigma)}}{(y - x) \dots (yv^n - xu^n)} \Bigg|_{(x/y)^k},$$

where $\text{expression}|_{t^j}$ means to select the coefficient of t^j in expression . Therefore, we have

$$\begin{aligned} & \sum_{n \geq 0} \frac{y t^n}{[n]_{p,q}!(y, x; v, u)_{n+1}} \sum_{\sigma \in S_n} x^{\text{des}(\sigma)} y^{\text{ris}(\sigma)} u^{\text{maj}(\sigma)} v^{\text{comaj}(\sigma)} q^{\text{inv}(\sigma)} p^{\text{coinv}(\sigma)} \\ &= \sum_{k \geq 0} (x/y)^k \zeta \left(\sum_{n \geq 0} t^n h_n \right) \\ &= \sum_{k \geq 0} (x/y)^k \left(\sum_{n \geq 0} (-t)^n \zeta(e_n) \right)^{-1} \\ &= \sum_{k \geq 0} (x/y)^k \left(\sum_{n \geq 0} (-t)^n \sum_{\substack{i_0, \dots, i_k \geq 0 \\ i_0 + \dots + i_k = n}} \frac{(u/v)^{0i_0 + \dots + ki_k}}{[i_0]_{p,q}! \dots [i_k]_{p,q}!} q^{\binom{i_0}{2} + \dots + \binom{i_k}{2}} \right)^{-1}, \end{aligned}$$

which may be simplified to equal the desired expression. \square

Theorem 2 was proved in a paper of Garsia and Gessel by playing with the set of nonnegative integer valued functions on the first n integers [8]. Without their work, the homomorphism ζ would have been more difficult to find.

The proof we have given for Theorem 2, although elementary and combinatorial, is not any “easier” than that given by Garsia and Gessel. However, there are at least two distinct advantages of our methods. First, the techniques in the

proof of Theorem 2 may be slightly modified to give a wide swath of seemingly unrelated generating functions for the permutation enumeration of the symmetric group, Weyl groups of type B and D , subsets of the symmetric group, the exponential formula, linear recurrence equations, and more [12]. Second, the ideas in the proof of Theorem 2 may be generalized to give new generating functions involving the descent, major index, and inversion statistics. We explore these ideas in the next section.

3. Generating functions for wreath product groups and multiples of permutations

Let us first turn our attention to applying this machinery to the hyperoctahedral group B_n and its subgroup D_n . The hyperoctahedral group B_n may be considered the set of permutations of n where each integer in the permutation is assigned either a $+$ or $-$ sign. For $\sigma \in B_n$, let $\text{pos}(\sigma)$ count the total number of positive signs and $\text{neg}(\sigma)$ count the total number of negative signs in σ . The subgroup D_n of B_n contains those $\sigma \in B_n$ with $\text{neg}(\sigma)$ an even number. These are Weyl groups appearing in the study of root systems and Lie algebras.

Define a linear order Θ on $\{\pm 1, \dots, \pm n\}$ such that

$$1 <_{\Theta} \dots <_{\Theta} n <_{\Theta} -n <_{\Theta} \dots <_{\Theta} -1.$$

This definition of the linear order arises from an interpretation of B_n as a Coxeter group, but any linear order can be made to work in the same ways in which Θ will be used. Define $\text{des}_B(\sigma)$ on B_n such that

$$\text{des}_B(\sigma) = \sum_i \chi(\sigma_{i+1} <_{\Theta} \sigma_i).$$

For $\sigma \in B_n$, let $\text{ris}_B(\sigma)$ be the statistic with the property that $\text{des}_B(\sigma) + \text{ris}_B(\sigma) = n$. Take $\text{inv}_B(\sigma)$ and $\text{coinv}_B(\sigma)$ be the inversion and co-inversion statistics with respect to the linear order Θ . Let $\text{maj}_B(\sigma)$ and $\text{comaj}_B(\sigma)$ be the major index and co-major index statistics for B_n defined such that $\text{maj}_B(\sigma)$ sums i if index i satisfies $\sigma_{i+1} <_{\Theta} \sigma_i$ and $\text{comaj}_B(\sigma)$ satisfies $\text{maj}_B(\sigma) + \text{comaj}_B(\sigma) = \binom{n+1}{2}$. All of these statistics collapse to their counterparts for the symmetric group when considering the subgroup of B_n isomorphic to S_n found by only taking permutations with integers associated with a positive sign.

Define a homomorphism ζ_B on the ring of symmetric functions by defining it on e_n such that

$$\zeta_B(e_n) = \sum_{\substack{i_0, \dots, i_k \geq 0 \\ i_0 + \dots + i_k = n}} \frac{(u/v)^{0i_0 + \dots + ki_k}}{[i_0]_{p,q}! \cdots [i_k]_{p,q}!} q^{\binom{i_0}{2} + \dots + \binom{i_k}{2}} [i_0 + 1]_{w,z} \cdots [i_k + 1]_{w,z}.$$

This is the homomorphism ζ with the addition of the factor of the form $[i_0 + 1]_{w,z} \cdots [i_k + 1]_{w,z}$. This factor will allow us to designate some cells as positive while others negative. In the positive cells we will insert a w while in the negative ones we will insert z .

Theorem 3.

$$\begin{aligned} & \sum_{n \geq 0} \frac{t^n}{[n]_{p,q}!(y, x; v, u)_{n+1}} \sum_{\sigma \in B_n} x^{\text{des}_B(\sigma)} y^{\text{ris}_B(\sigma)} u^{\text{maj}_B(\sigma)} v^{\text{comaj}_B(\sigma)} q^{\text{inv}_B(\sigma)} p^{\text{coinv}_B(\sigma)} w^{\text{pos}(\sigma)} z^{\text{neg}(\sigma)} \\ &= \sum_{k \geq 0} \frac{x^k (w - z)^k}{y^{k+1} (w \mathbf{e}_{p,q}^{-tw(u/v)^0} - z \mathbf{e}_{p,q}^{-tz(u/v)^0}) \cdots (w \mathbf{e}_{p,q}^{-tw(u/v)^k} - z \mathbf{e}_{p,q}^{-tz(u/v)^k})}. \end{aligned}$$

Proof. Since the only difference between the definitions of ζ and ζ_B is the inclusion of a factor of the form $[i_0 + 1]_{w,z} \cdots [i_k + 1]_{w,z}$, we may form brick tabloids from expanding $[n]_{p,q}! \zeta_B(h_n)$ in terms of the elementary symmetric functions in a similar way as found in the proof of Theorem 2.

−1	−1	−1	−1	1	−1	1	−1	−1	1	1	1
z	w	z	z	w	w	z	w	z	w	z	w
u^0v^0	u^0v^0	u^1v^{-1}	u^1v^{-1}	u^1v^{-1}	u^2v^{-2}	u^3v^{-3}	u^1v^{-1}	u^3v^{-3}	u^3v^{-3}	u^0v^0	u^1v^{-1}
$q^{11}p^0$	q^0p^{10}	q^8p^1	q^7p^1	q^0p^7	q^1p^5	q^0p^5	q^1p^3	q^1p^3	q^0p^2	q^1p^0	q^0p^0
12	1	10	9	2	4	3	6	7	5	11	8

Fig. 6. An example of a combinatorial object coming from the application of ξ_B to $[n]_{p,q}!h_n$.

−1	−1	−1	−1	1	−1	1	−1	−1	1	1	1
z	w	z	z	w	w	z	w	z	w	z	w
u^0v^0	u^0v^0	u^1v^{-1}	u^1v^{-1}	u^1v^{-1}	u^2v^{-2}	u^3v^{-3}	u^1v^{-1}	u^3v^{-3}	u^3v^{-3}	u^0v^0	u^1v^{-1}
$q^{11}p^0$	q^0p^{10}	q^8p^1	q^7p^1	q^0p^7	q^1p^5	q^0p^5	q^1p^3	q^1p^3	q^0p^2	q^1p^0	q^0p^0
12	1	9	10	2	4	3	6	7	5	11	8

Fig. 7. Reversing the order of two integers in Fig. 6.

Let us start with a brick tabloid as described in the proof of Theorem 2 and displayed in Figs. 3–5. The powers of w and z appear in the form

$$[i_j + 1]_{w,z} = w^{i_j} z^0 + \cdots + w^0 z^{i_j}.$$

This allows us to select one monomial in w and z of degree i_j for each string of consecutive cells in a brick labeled with the same power of (u/v) . With this choice, place a string of z 's in the first cells and w 's in the remaining. This filling of brick tabloids, together with the filling described in the proof of Theorem 2, gives objects whose weighted sum is equal to the application of ξ_B to $[n]_{p,q}!h_n$. An example of one such brick tabloid may be found in Fig. 6.

Suppose we have j consecutive cells within a brick with the same power of (u/v) and marked with a z . Instead of writing these integers in decreasing order as prescribed in the proof of Theorem 2, let us reverse the order of these j integers so that they are in increasing order. This is done so that cells in a brick marked with the same power of (u/v) are in descending order according to the linear order Θ . For instance, the result of performing this switch to the object in Fig. 6 may be found in Fig. 7. The same brick breaking/combining sign-reversing weight-preserving involution as in Theorem 2 may be now applied to leave a set of fixed points which may be seen to correspond to elements in B_n . In fact, all of the proof of Theorem 2 follows through in the exact same manner in this case. By employing (2), the generating function for

$$\sum_{n \geq 0} \frac{y t^n}{[n]_{p,q}!(y, x; v, u)_{n+1}} \sum_{\sigma \in B_n} x^{\text{des}_B(\sigma)} y^{\text{ris}_B(\sigma)} u^{\text{maj}_B(\sigma)} v^{\text{comaj}_B(\sigma)} q^{\text{inv}_B(\sigma)} p^{\text{coinv}_B(\sigma)} w^{\text{pos}(\sigma)} z^{\text{neg}(\sigma)}$$

is equal to

$$\sum_{k \geq 0} (x/y)^k \left(\sum_{n \geq 0} (-t)^n \sum_{\substack{i_0, \dots, i_k \geq 0 \\ i_0 + \dots + i_k = n}} \frac{(u/v)^{0i_0 + \dots + ki_k}}{[i_0]_{p,q}! \cdots [i_k]_{p,q}!} q^{\binom{i_0}{2} + \dots + \binom{i_k}{2}} [i_0 + 1]_{w,z} \cdots [i_k + 1]_{w,z} \right)^{-1}.$$

which in turn may be simplified to look like the statement of the theorem. \square

As for D_n , the subgroup consisting of those $\sigma \in B_n$ with $\text{neg}(\sigma)$ an even number, notice that

$$\frac{1}{2} z^{\text{neg}(\sigma)} + \frac{1}{2} (-z)^{\text{neg}(\sigma)} = \begin{cases} z^{\text{neg}(\sigma)} & \text{if } \sigma \in D_n, \\ 0 & \text{if } \sigma \notin D_n. \end{cases} \quad (6)$$

Therefore, the generating function in Theorem 3 may be specialized for D_n . We have

$$\begin{aligned} & \sum_{n \geq 0} \frac{t^n}{[n]_{p,q}!(y, x; v, u)_{n+1}} \sum_{\sigma \in D_n} x^{\text{des}_B(\sigma)} y^{\text{ris}_B(\sigma)} u^{\text{maj}_B(\sigma)} v^{\text{comaj}_B(\sigma)} q^{\text{inv}_B(\sigma)} p^{\text{coinv}_B(\sigma)} w^{\text{pos}(\sigma)} z^{\text{neg}(\sigma)} \\ &= \frac{1}{2} \sum_{k \geq 0} \frac{x^k (w - z)^k}{y^{k+1} (w \mathbf{e}_{p,q}^{-tw(u/v)^0} - z \mathbf{e}_{p,q}^{-tz(u/v)^0}) \cdots (w \mathbf{e}_{p,q}^{-tw(u/v)^k} - z \mathbf{e}_{p,q}^{-tz(u/v)^k})} \\ &+ \frac{1}{2} \sum_{k \geq 0} \frac{x^k (w + z)^k}{y^{k+1} (w \mathbf{e}_{p,q}^{-tw(u/v)^0} + z \mathbf{e}_{p,q}^{tz(u/v)^0}) \cdots (w \mathbf{e}_{p,q}^{-tw(u/v)^k} + z \mathbf{e}_{p,q}^{tz(u/v)^k})}. \end{aligned}$$

The same labeling scheme applied to brick tabloids to find results for the hyperoctahedral group work for more general wreath product groups (of which the hyperoctahedral group is a special case). Let G be a finite group. The group $G \wr S_n$ is defined as

$$G \wr S_n = \{(f, \sigma) \mid f : \{1, \dots, n\} \rightarrow G \text{ and } \sigma \in S_n\}$$

and is referred to as the wreath product of G with S_n . A convenient way to think of the group is by considering the set of $n \times n$ permutation matrices where each 1 is an element of G . In this light, group multiplication is defined to be matrix multiplication. Elements in $G \wr S_n$ can be presented in matrix or one line notation where each integer in a permutation in S_n is paired with an element in G . For example, if g_1, \dots, g_5 are in G , $\sigma \in G \wr S_n$ may be given by

$$\sigma = \begin{pmatrix} 0 & g_2 & 0 & 0 & 0 \\ g_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & g_4 & 0 \\ 0 & 0 & g_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & g_5 \end{pmatrix}$$

or

$$\sigma = (g_2, 1) (g_1, 2) (g_4, 3) (g_3, 4) (g_5, 5).$$

From here it may be seen that the hyperoctahedral group is the special case of the wreath product group $\mathbb{Z}_2 \wr S_n$.

Arbitrarily fix some linear order among elements $\sigma \in G \wr S_n$. According to this linear order, a version of descents, major index, and inversion statistics may be defined using this linear order in an analogous manner as we have already done in the case of the hyperoctahedral group. Using the same techniques as in the proof of Theorem 3, a generating function registering these statistics over $G \wr S_n$ —together with a count on how many times a specific group element is paired with an integer—may be easily found. All that is needed is a slight modification of the homomorphism ξ_B .

Specifically, the difference of the homomorphisms ξ and ξ_B are the factors of the form $[i_j + 1]_{w,z}$. These factors designated each cell in a brick tabloid as either positive or negative. In the case of wreath product groups, if $|G| = m$, we would need to change this factor to something like

$$\sum_{a_1 + \cdots + a_m = i_j} w_1^{a_1} \cdots w_m^{a_m}$$

so that the indeterminates w_1, \dots, w_m would keep track of the number of times a specific group element is paired with an integer in the wreath product group. Note that this factor is actually the i_j th homogeneous symmetric function $h_{i_j}(w_1, \dots, w_m, 0, 0, \dots)$.

The same type of tricks that gave a generating function for D_n can be used for other wreath product groups. Since we can keep track of which group elements are used to pair with integers in $\sigma \in G \wr S_n$, we can use this information to restrict our generating functions to specific subgroups. For example, consider the wreath product group $\{1, \varepsilon, \dots, \varepsilon^{k-1}\} \wr S_n$ where $\varepsilon = e^{(2\pi i)/k}$ is a primitive k th root of unity (this is the group $\mathbb{Z}_k \wr S_n$ where we are representing \mathbb{Z}_k as a multiplicative group). Define the subgroup $D_{k,n}$ as

$$D_{k,n} = \{\sigma \in \{1, \dots, \varepsilon^{k-1}\} \wr S_n : \text{the product of the group elements paired with the integers in } \sigma \text{ is } 1\}.$$

Suppose that $\text{neg}_i(\sigma)$ is the statistic counting the number of integers in $\sigma \in \{1, \dots, \varepsilon^{k-1}\} \wr S_n$ paired with ε^i for $i = 0, \dots, k-1$. It follows that

$$\frac{1}{k} ((\varepsilon^{0\text{neg}_0(\sigma)} \dots \varepsilon^{(k-1)\text{neg}_{k-1}(\sigma)})^0 + \dots + (\varepsilon^{0\text{neg}_0(\sigma)} \dots \varepsilon^{(k-1)\text{neg}_{k-1}(\sigma)})^{k-1})$$

is equal to 1 if $\sigma \in D_{k,n}$ and 0 otherwise. This fact may be used in the same manner as (6) to restrict the generating function for permutation statistics over $\mathbb{Z}_k \wr S_n$ to the subgroup $D_{k,n}$ as we have done for $D_n \subset \mathbb{Z}_2 \wr S_n$. To display one such generating function, the relationship

$$\frac{1}{3} (1 + \varepsilon^{\text{neg}_1(\sigma)} (\varepsilon^2)^{\text{neg}_2(\sigma)} + (\varepsilon^2)^{\text{neg}_1(\sigma)} \varepsilon^{\text{neg}_2(\sigma)}) = \begin{cases} 1 & \text{if } \sigma \in D_{3,n}, \\ 0 & \text{if } \sigma \notin D_{3,n}, \end{cases}$$

where $\varepsilon^3 = 1$ may be used to find that

$$\begin{aligned} & \sum_{n \geq 0} \frac{t^n}{[n]_{p,q}!(y, x; v, u)_{n+1}} \sum_{\sigma \in D_{3,n}} x^{\text{des}_{\mathbb{Z}_3 \wr S_n}(\sigma)} y^{\text{ris}_{\mathbb{Z}_3 \wr S_n}(\sigma)} u^{\text{maj}_{\mathbb{Z}_3 \wr S_n}(\sigma)} v^{\text{comaj}_{\mathbb{Z}_3 \wr S_n}(\sigma)} q^{\text{inv}_{\mathbb{Z}_3 \wr S_n}(\sigma)} q^{\text{coinv}_{\mathbb{Z}_3 \wr S_n}(\sigma)} \\ &= \frac{1}{3} \sum_{k \geq 0} \frac{x^k}{y^{k+1}} \left(\sum_{n=0}^{\infty} (-t)^n \sum_{i_0+\dots+i_k=n} \frac{(u/v)^{0i_0+\dots+ki_k} q^{\binom{i_0}{2}+\dots+\binom{i_k}{2}} h_{i_0}(1, 1, 1) \dots h_{i_k}(1, 1, 1)}{[i_0]_{p,q}! \dots [i_k]_{p,q}!} \right)^{-1} \\ &+ \frac{1}{3} \sum_{k \geq 0} \frac{x^k}{y^{k+1}} \left(\sum_{n=0}^{\infty} (-t)^n \sum_{i_0+\dots+i_k=n} \frac{(u/v)^{0i_0+\dots+ki_k} q^{\binom{i_0}{2}+\dots+\binom{i_k}{2}} h_{i_0}(1, \varepsilon, \varepsilon^2) \dots h_{i_k}(1, \varepsilon, \varepsilon^2)}{[i_0]_{p,q}! \dots [i_k]_{p,q}!} \right)^{-1} \\ &+ \frac{1}{3} \sum_{k \geq 0} \frac{x^k}{y^{k+1}} \left(\sum_{n=0}^{\infty} (-t)^n \sum_{i_0+\dots+i_k=n} \frac{(u/v)^{0i_0+\dots+ki_k} q^{\binom{i_0}{2}+\dots+\binom{i_k}{2}} h_{i_0}(1, \varepsilon^2, \varepsilon) \dots h_{i_k}(1, \varepsilon^2, \varepsilon)}{[i_0]_{p,q}! \dots [i_k]_{p,q}!} \right)^{-1} \end{aligned}$$

where $\text{stat}_{\mathbb{Z}_2 \wr S_n}(\sigma)$ is the statistic found when using the linear order

$$(1, 1) < \dots < (1, n) < (\varepsilon, 1) < \dots < (\varepsilon, n) < (\varepsilon^2, 1) < \dots < (\varepsilon^2, n).$$

There are other groups for which this trick may be applied—all that is needed is some relationship between the group elements which afford an expression like (6).

Besides the cases for wreath product groups and their subgroups, the proof of Theorem 2 may be applied to find new results about multiples of permutations. By this we simply mean we can find permutation statistics for the m -fold product of the symmetric group S_n . Let the Cartesian product $S_n \times \dots \times S_n$ (m times) be denoted by S_n^m . For $(\sigma^1, \dots, \sigma^m) \in S_n^m$, let $\text{comdes}(\sigma^1, \dots, \sigma^m)$ be the statistic counting the number of times σ^i has a descent occurring at the j th place for every i . This is known as the number of common descents of $(\sigma^1, \dots, \sigma^m)$. Let $\text{commaj}(\sigma^1, \dots, \sigma^m)$ be the common major index counting i every time $\sigma_i^j > \sigma_{i+1}^j$ for $j = 1, \dots, m$.

If we define a homomorphism ξ_m mapping e_n to

$$\sum_{\substack{i_0, \dots, i_k \geq 0 \\ i_0 + \dots + i_k = n}} \frac{u^{0i_0 + \dots + ki_k}}{([i_0]_{p_1, q_1}! \dots [i_0]_{p_m, q_m}!) \dots ([i_k]_{p_1, q_1}! \dots [i_k]_{p_m, q_m}!)} (q_1 \dots q_m)^{\binom{i_0}{2} + \dots + \binom{i_k}{2}},$$

then a generating function for

$$\begin{aligned} & \sum_{n \geq 0} \frac{t^n}{[n]_{p_1, q_1}! \dots [n]_{p_m, q_m}! (1, x; 1, u)_{n+1}} \\ & \times \sum_{(\sigma^1, \dots, \sigma^m) \in S_n^m} x^{\text{comdes}(\sigma^1, \dots, \sigma^m)} u^{\text{commaj}(\sigma^1, \dots, \sigma^m)} q_1^{\text{inv}(\sigma^1)} p_1^{\text{inv}(\sigma^1)} \dots q_m^{\text{inv}(\sigma^m)} p_m^{\text{inv}(\sigma^m)} \end{aligned}$$

−1	−1	−1	−1	1	−1	1	−1	−1	1	1	1
u^0	u^0	u^1	u^1	u^1	u^2	u^3	u^1	u^3	u^3	u^0	u^1
$q_1^{11}p_1^0$	$q_1^0p_1^{10}$	$q_1^8p_1^1$	$q_1^7p_1^1$	$q_1^0p_1^7$	$q_1^1p_1^5$	$q_1^0p_1^5$	$q_1^1p_1^3$	$q_1^1p_1^3$	$q_1^0p_1^2$	$q_1^1p_1^0$	$q_1^0p_1^0$
12	1	10	9	2	4	3	6	7	5	11	8
$q_2^3p_2^8$	$q_2^2p_2^8$	$q_2^2p_2^7$	$q_2^8p_2^0$	$q_2^0p_2^7$	$q_2^6p_2^0$	$q_2^0p_2^5$	$q_2^1p_2^3$	$q_2^1p_2^2$	$q_2^0p_2^2$	$q_2^1p_2^0$	$q_2^0p_2^0$
4	3	12	10	1	11	2	6	8	5	9	7

Fig. 8. A decorated brick tabloid in the case of multiples of permutations when $m = 2$.

is equal to

$$\sum_{k \geq 0} x^k \left(\sum_{n \geq 0} (-t)^n \zeta_m(e_n) \right)^{-1}.$$

The proof of this fact hinges upon the labeling of brick tabloids with many different permutations instead of just one. Along with each of these permutations powers of q_i and p_i are assigned to keep track of inversions and coinversions. For example, one such brick tabloid when $m=2$ may be found in Fig. 8. This labeling is afforded by the homomorphism ζ_m ; in fact, the definition of the homomorphism ζ_m was specifically designed to produce these objects. Note how every p , q -analogue in the definition of ζ has been written down m times to give rise to the m rows of permutations in a brick tabloid.

The sign-reversing weight-preserving involution in the proof of Theorem 2 must be changed slightly to accommodate these objects. Bricks should be scanned from left to right looking for a cell containing -1 or two consecutive bricks which may be combined. Fixed points under this involution are the same as in Fig. 5 with the exception that if two consecutive bricks have the same power of (u/v) , then at least one of the permutations must increase there. The rest of the calculations to show the desired result work through in the precise same way as in Section 2.

We can combine the labeling schemes in the wreath product case together with the ideas which gave the result for multiples of wreath product groups. We will give an example of this in the next section.

4. More generating functions for wreath product groups

One of the goals of this section is to find generating functions which include a modified version of the descent statistic defined on the hyperoctahedral group B_n . That is, for $\sigma \in B_n$, we want to understand the statistic

$$\chi(n <_{\Theta} \sigma_n) + \sum_i \chi(\sigma_{i+1} <_{\Theta} \sigma_i).$$

This is the same as $\text{des}_B(\sigma)$ with the exception of the term of the form $\chi(n <_{\Theta} \sigma_n)$. There are two reasons our attention turns to this statistic. First, it has already appeared in the literature. Using the methods of Garsia and Gessel and the study of upper binomial posets, Reiner found a generalization of Theorem 2 which included this statistic [13]. Second, it displays the versatility of our methods and explicitly shows how our methods can be modified so that other generation functions can easily be found.

We will define a ring homomorphism on the elementary symmetric functions and apply this homomorphism to a different basis in the ring of symmetric functions. Brick tabloids will be decorated and weighted in a specific manner. Then, a sign-reversing weight-preserving involution will be applied leaving fixed points corresponding to permutations in B_n weighted with appropriate powers of indeterminates. The major difference between the proofs in this section, however, is that we will not apply our homomorphisms to the homogeneous symmetric functions. Instead, we define a new class of symmetric functions on which to apply our homomorphisms.

Let v be some function on the set of nonnegative integers. Define $p_{n,v} \in A_n$ recursively such that

$$p_{n,v} = (-1)^{n-1} v(n) e_n + \sum_{k=1}^{n-1} (-1)^{k-1} e_k p_{n-k,v}$$

for all $n \geq 1$. This means that

$$\begin{aligned} E(-t) \sum_{n \geq 1} p_{n,v} t^n &= \left(\sum_{n \geq 0} (-1)^n e_n t^n \right) \left(\sum_{n \geq 1} p_{n,v} t^n \right) \\ &= \sum_{n \geq 1} \left(\sum_{k=0}^{n-1} p_{n-k,v} (-1)^k e_k \right) t^n \\ &= \sum_{n \geq 1} (-1)^{n-1} v(n) e_n t^n, \end{aligned}$$

where the last equality follows from the definition of $p_{n,v}$. Therefore,

$$\sum_{n \geq 1} p_{n,v} t^n = \frac{\sum_{n \geq 1} (-1)^{n-1} v(n) e_n t^n}{E(-t)} = \frac{\sum_{n \geq 1} (-1)^{n-1} v(n) e_n t^n}{\sum_{n \geq 0} (-1)^n e_n t^n}. \quad (7)$$

Notice that by taking $v(n) = 1$ for all $n \geq 1$, we can use the above equation to show that

$$1 + \sum_{n \geq 1} p_{n,1} t^n = 1 + \frac{\sum_{n \geq 1} (-1)^{n-1} e_n t^n}{\sum_{n \geq 0} (-1)^n e_n t^n} = \frac{1}{\sum_{n \geq 0} (-1)^n e_n t^n} = \sum_{n \geq 0} h_n t^n.$$

This means that $p_{n,1} = h_n$. Other choices of v give well known generating functions. For example, by taking v such that $v(n) = n$ for all $n \geq 1$, it may be shown that $p_{n,n}$ is the power symmetric function equal to $\sum_i x_i^n$. Further, by taking $v(n) = (-1)^k \chi(n \geq k+1)$ for some $k \geq 1$, one may show that $p_{n,(-1)^k \chi(n \geq k+1)}$ is the Schur function corresponding to the shape $(1^k, n)$.

This definition of $p_{n,v}$ is desirable because of its expansion in terms of the elementary symmetric functions. The coefficient of e_λ in $p_{n,v}$ has a nice combinatorial interpretation in terms of brick tabloids where an extra weight is associated to each tabloid. Suppose T is a brick tabloid of shape (n) and type λ and that the final brick in T has length ℓ . Define $w_v(T)$ to be equal to $v(\ell)$ and let $w_v(B_{\lambda,n})$ be the sum of weights of all brick tabloids T of shape (n) and type λ . The number of brick tabloids $B_{\lambda,n}$ we have used in previous sections is the special case $v(n) = 1$. By the recursions found in the definition of $p_{n,v}$, it may be shown that

$$p_{n,v} = \sum_{\lambda \vdash n} (-1)^{n-\ell(\lambda)} w_v(B_{\lambda,n}) e_\lambda.$$

in almost the exact same way that (3) above was proved in [7].

At this point we have the two necessary expressions which gave our generating functions in the previous section: a generating function expressing $p_{n,v}$ in terms of the elementary symmetric functions, and a combinatorial expansion of the coefficient of e_λ in $p_{n,v}$ in terms of brick tabloids.

Define a function v_1 such that

$$v_1(n) = \frac{\sum_{\substack{i_0, \dots, i_k \geq 0 \\ i_0 + \dots + i_k = n}} \frac{(u/v)^{0i_0 + \dots + ki_k}}{[i_0]_{p,q}! \cdots [i_k]_{p,q}!} q^{\binom{i_0}{2} + \dots + \binom{i_k}{2}} [i_0 + 1]_{w,z} \cdots [i_{k-1} + 1]_{w,z} ([i_k + 1]_{w,z} + (s-1)z^{i_k})}{\sum_{\substack{i_0, \dots, i_k \geq 0 \\ i_0 + \dots + i_k = n}} \frac{(u/v)^{0i_0 + \dots + ki_k}}{[i_0]_{p,q}! \cdots [i_k]_{p,q}!} q^{\binom{i_0}{2} + \dots + \binom{i_k}{2}} [i_0 + 1]_{w,z} \cdots [i_k + 1]_{w,z}}.$$

which, although it may appear a horrible cacophony of symbols, is designed to interact nicely with the homomorphism ξ_B .

Theorem 4.

$$\begin{aligned}
& \sum_{n \geq 1} \frac{t^n}{[n]_{p,q}!(y, x; v, u)_{n+1}} \sum_{\sigma \in B_n} x^{\text{des}_B(\sigma)} y^{\text{ris}_B(\sigma)} u^{\text{maj}_B(\sigma)} v^{\text{comaj}_B(\sigma)} q^{\text{inv}_B(\sigma)} p^{\text{coinv}_B(\sigma)} w^{\text{pos}(\sigma)} z^{\text{neg}(\sigma)} s^{\chi(n < \theta \sigma_n)} \\
&= \sum_{k \geq 0} \frac{x^k (w - z)^k}{y^{k+1} (w \mathbf{e}_{p,q}^{-tw(u/v)^0} - z \mathbf{e}_{p,q}^{-tz(u/v)^0}) \cdots (w \mathbf{e}_{p,q}^{-tw(u/v)^k} - z \mathbf{e}_{p,q}^{-tz(u/v)^k})} \\
&\quad - \sum_{k \geq 0} \frac{x^k (w - z)(s - 1) \mathbf{e}_{p,q}^{-tz(u/v)^k}}{y^{k+1} (w \mathbf{e}_{p,q}^{-tw(u/v)^k} - z \mathbf{e}_{p,q}^{-tz(u/v)^k})}.
\end{aligned}$$

Proof. First we apply ζ_B to $[n]_{p,q}! p_{n,v_1}$. We have

$$[n]_{p,q}! \zeta(p_{n,v_1}) = [n]_{p,q}! \sum_{\lambda \vdash n} (-1)^{n-\ell(\lambda)} w_{v_1}(B_{\lambda,n}) \zeta_B(e_\lambda).$$

The right-hand side of the above equation is the same as (4) except that instead of “regular” brick tabloids we are dealing with weighted brick tabloids. Therefore, we can construct brick tabloids filled with indeterminates much in the same way as in the proof of Theorem 3. The only difference between the objects in the proof of Theorem 3 and our current situation is the last brick weight. Thus, let us start with a brick tabloid T which is filled with a permutation and weighted with powers of p, q, u, v, w, z and -1 just as found in Fig. 7 and described in the proof of Theorem 3.

Suppose that the last brick in T has length ℓ meaning that we have a $v_1(\ell)$ term to assign to this brick tabloid. Now, one factor of the form $\zeta_B(e_\ell)$ appears in the denominator of $v_1(\ell)$. This allows us to erase all of the weights and labels in the final brick of T which came from $\zeta_B(e_\ell)$. The final brick can now be filled with the numerator in the definition of $v_1(\ell)$.

This numerator of $v_1(\ell)$ is the exact same object as $\zeta_B(e_\ell)$ with one difference. The term of the form $[i_k + 1]_{w,z}$ in the summand is replaced with

$$[i_k + 1]_{w,z} + (s - 1)z^{i_k} = w^{i_k} z^0 + \cdots + w^1 z^{i_k-1} + s w^0 z^{i_k}.$$

Thus, we can fill the final brick in T in the same way as the rest of the bricks except that if every cell in the final brick is assigned a z , an s is written in the terminal cell of the brick tabloid. Fig. 9 is an example of one brick tabloid T which counts one object coming from $[n]_{p,q}! \zeta_B(p_{n,v})$. It follows that the weighted sum over all possible brick tabloids T created in this manner is equal to $[n]_{p,q}! \zeta_B(p_{n,v})$.

The same brick breaking/combining weight-preserving, sign-reversing involution as found in the proof of Theorem 3 may be now be applied to this collection of brick tabloids. Counting the fixed points left after an application of this involution, it may be seen in the same way as in the proof of Theorem 2 that $[n]_{p,q}! \zeta_B(p_{n,v})$ is equal to

$$\frac{1}{(y, x; v, u)_{n+1}} \sum_{\sigma \in B_n} x^{\text{des}_B(\sigma)} y^{\text{ris}_B(\sigma)+1} u^{\text{maj}_B(\sigma)} v^{\text{comaj}_B(\sigma)} q^{\text{inv}_B(\sigma)} p^{\text{coinv}_B(\sigma)} s^{\chi(n < \theta \sigma_n)} \bigg|_{(x/y)^k}.$$

Therefore,

$$\begin{aligned}
& \sum_{n \geq 1} \frac{y t^n}{[n]_{p,q}!(y, x; v, u)_{n+1}} \sum_{\sigma \in B_n} x^{\text{des}_B(\sigma)} y^{\text{ris}_B(\sigma)} u^{\text{maj}_B(\sigma)} v^{\text{comaj}_B(\sigma)} q^{\text{inv}_B(\sigma)} p^{\text{coinv}_B(\sigma)} s^{\chi(n < \theta \sigma_n)} \\
&= \sum_{k \geq 0} (x/y)^k \zeta_B \left(\sum_{n \geq 0} t^n p_{n,v_1} \right) \\
&= \sum_{k \geq 0} (x/y)^k \left(\frac{\sum_{n \geq 1} (-1)^{n-1} v_1(n) \zeta_B(e_n) t^n}{\sum_{n \geq 0} (-1)^n \zeta_B(e_n) t^n} \right),
\end{aligned}$$

which, by expanding the definitions of $v_1(n)$ and $\zeta_B(e_n)$ and simplifying may be seen to equal the statement of the theorem. \square

−1	−1	−1	−1	1	−1	1	−1	−1	1	1	1
z	w	z	z	w	w	z	w	z	w	z	sz
u^0v^0	u^0v^0	u^1v^{-1}	u^1v^{-1}	u^1v^{-1}	u^2v^{-2}	u^3v^{-3}	u^1v^{-1}	u^3v^{-3}	u^3v^{-3}	u^0v^0	u^1v^{-1}
$q^{11}p^0$	q^0p^{10}	q^8p^1	q^7p^1	q^0p^7	q^1p^5	q^0p^5	q^1p^3	q^1p^3	q^0p^2	q^1p^0	q^0p^0
12	1	9	10	2	4	3	6	7	5	8	11

Fig. 9. A brick tabloid with the final brick altered to include a power of s .

−1	−1	−1	−1	1	−1	1	−1	−1	1	−1	1
z_1z_2	z_1w_2	w_1w_2	w_1w_2	w_1w_2	w_1z_2	z_1w_2	w_1z_2	z_1z_2	w_1w_2	z_1z_2	$s_1z_1w_2$
u^0	u^0	u^1	u^1	u^1	u^2	u^3	u^1	u^3	u^3	u^0	u^1
$q_1^{11}p_1^0$	$q_1^0p_1^{10}$	$q_1^8p_1^1$	$q_1^7p_1^1$	$q_1^0p_1^7$	$q_1^1p_1^5$	$q_1^0p_1^5$	$q_1^1p_1^3$	$q_1^1p_1^3$	$q_1^0p_1^2$	$q_1^1p_1^0$	$q_1^0p_1^0$
1	12	9	10	2	4	3	6	7	5	8	11
$q_2^3p_2^8$	$q_2^2p_2^8$	$q_2^2p_2^7$	$q_2^8p_2^0$	$q_2^0p_2^7$	$q_2^6p_2^0$	$q_2^0p_2^5$	$q_2^1p_2^3$	$q_2^1p_2^2$	$q_2^0p_2^2$	$q_2^1p_2^0$	$q_2^0p_2^0$
4	3	10	12	1	11	2	6	8	5	9	7

Fig. 10. A brick tabloid decorated to count elements in $B_n \times B_n$.

The methods in the proof of this theorem indicate that, in general, if we know a generating function for a statistic on a permutation group, then the net result of learning about the last integer in the permutation is an extra term in the numerator of the generating function.

As a final example, we will give a generating function for $B_n \times B_n$ where we keep track of the final integer in each of the two elements in B_n separately. If we would like to find a generating function for $B_n \times B_n$ which registers common descents, common major index, and inversions for the two different permutations with respect to the linear order θ along with a count if the final sign in each of the two permutations is negative, we can decorate brick tabloids like that in Fig. 10 below. These may be seen to correspond to elements in $B_n \times B_n$: there are factors of z_1 or w_1 and z_2 or w_2 to associate either a positive or negative sign to integers in both permutations written in the brick tabloid. We will place a power of s_1 or s_2 or both in the final cell of the brick tabloid whenever the final cell contains either z_1 or z_2 , respectively. These factors of s_1 and s_2 will count if the final cell corresponds to a positive integer or a negative integer.

In order to construct the object found in Fig. 10, we will apply a homomorphism to the basis $p_{n,v}$ for some function v . The “double” version of the homomorphism ξ_B which will give the labeling of all but the final brick is defined by

$$\xi_{B,2}(e_n) = \sum_{i_0+\dots+i_k=n} \frac{u^{0i_0+\dots+ki_k} q_1^{\binom{i_0}{2}+\dots+\binom{i_k}{2}} q_2^{\binom{i_0}{2}+\dots+\binom{i_k}{2}}}{[i_0]_{p_1,q_1}! [i_0]_{p_2,q_2}! \cdots [i_k]_{p_1,q_1}! [i_k]_{p_2,q_2}!} \times [i_0+1]_{w_1,z_1} [i_0+1]_{w_2,z_2} \cdots [i_k+1]_{w_1,z_1} [i_k+1]_{w_2,z_2}.$$

This is the homomorphism ξ_B with every term involving a power of q , p , w or z written down twice to produce the two elements in B_n as described in creating the generating function for multiples of permutations at the end of Section 3. To add the potential powers of s_1 and s_2 in the final brick in a brick tabloid like that displayed in Fig. 10, we need to determine the function v from which our new class of symmetric functions are created. Let us define v_2 such that $v_2(n)$ is equal to

$$\frac{\sum_{i_0+\dots+i_k=n} \frac{u^{0i_0+\dots+ki_k} q_1^{\binom{i_0}{2}+\dots+\binom{i_k}{2}} q_2^{\binom{i_0}{2}+\dots+\binom{i_k}{2}}}{[i_0]_{p_1,q_1}! [i_0]_{p_2,q_2}! \cdots [i_k]_{p_1,q_1}! [i_k]_{p_2,q_2}!} [i_0+1]_{w_1,z_1} [i_0+1]_{w_2,z_2} \cdots [i_{k-1}+1]_{w_1,z_1} [i_{k-1}+1]_{w_2,z_2} A}{\sum_{i_0+\dots+i_k=n} \frac{u^{0i_0+\dots+ki_k} q_1^{\binom{i_0}{2}+\dots+\binom{i_k}{2}} q_2^{\binom{i_0}{2}+\dots+\binom{i_k}{2}}}{[i_0]_{p_1,q_1}! [i_0]_{p_2,q_2}! \cdots [i_k]_{p_1,q_1}! [i_k]_{p_2,q_2}!} [i_0+1]_{w_1,z_1} [i_0+1]_{w_2,z_2} \cdots [i_k+1]_{w_1,z_1} [i_k+1]_{w_2,z_2}},$$

where A is equal to

$$([i_k+1]_{w_1,z_1} + (s_1-1)z_1^{i_k})([i_k+1]_{w_2,z_2} + (s_2-1)z_2^{i_k}).$$

Since we are dividing by $\zeta_{B,2}$ in the definition of v_2 , we are allowed to erase the labeling on the final brick in a brick tabloid and replace it with a labeling coming from the numerator of v_2 . The numerator in the definition of v_2 is designed to give the powers of s_1 and s_2 in the correct places. By our brick breaking/combining weight-preserving, sign-reversing involution and the other techniques permeating this work, it follows that:

$$\sum_{n \geq 1} \frac{t^n}{[n]_{q_1, p_1}! [n]_{q_2, p_2}! (1, x; 1, u)_{n+1}} \sum_{\sigma, \tau \in B_n} x^{\text{comdes}_B(\sigma, \tau)} u^{\text{commaj}_B(\sigma, \tau)} \\ \times q_1^{\text{inv}_B(\sigma)} p_1^{\text{coinv}_B(\sigma)} q_2^{\text{inv}_B(\tau)} p_2^{\text{coinv}_B(\tau)} w_1^{\text{pos}(\sigma)} z_1^{\text{neg}(\sigma)} w_2^{\text{pos}(\tau)} z_2^{\text{neg}(\tau)} s_1^{\chi(n <_{\theta} \sigma_n)} s_2^{\chi(n <_{\theta} \tau_n)}$$

is equal to

$$\sum_{k \geq 0} x^k \left(\frac{\sum_{n \geq 1} (-1)^{n-1} v_2(n) \zeta_{B,2}(e_n) t^n}{\sum_{n \geq 0} (-1)^n \zeta_{B,2}(e_n) t^n} \right).$$

In conclusion, we have given a new, combinatorial proof of a well-known formula of Ira Gessel. This new proof was easily modified to produce other generating functions for wreath product groups, subsets of wreath product groups, and multiples of permutations—some of which have appeared in less general form in the literature. There are many more generating functions which may be formed in this manner. Although some of them can become complicated, as our final example shows, they are all derived in the same way.

References

- [1] D. Beck, Permutation enumeration of the symmetric and hyperoctahedral group and the combinatorics of symmetric functions, Ph.D. Thesis, University of California, San Diego, 1993.
- [2] D. Beck, The combinatorics of symmetric functions and permutation enumeration of the hyperoctahedral group, *Discrete Math.* 163 (1997) 13–45.
- [3] D. Beck, J. Remmel, Permutation enumeration of the symmetric group and the combinatorics of symmetric functions, *J. Combin. Theory Ser. A* 72 (1995) 1–49.
- [4] F. Brenti, Unimodal polynomials arising from symmetric functions, *Proc. Amer. Math. Soc.* 108 (1990) 1133–1141.
- [5] F. Brenti, Permutation enumeration, symmetric functions, and unimodality, *Pacific J. Math.* 157 (1993) 1–28.
- [6] L. Carlitz, Sequences and inversions, *Duke Math. J.* 37 (1970) 193–198.
- [7] Ö. Eğecioğlu, J. Remmel, Brick tabloids and the connection matrices between bases of symmetric functions, *Discrete Appl. Math.* 34 (1991) 107–120.
- [8] A. Garsia, I. Gessel, Permutation statistics and partitions, *Adv. Math.* 31 (1979) 288–305.
- [9] I. Gessel, Generating functions and enumeration of sequences, Ph.D. Thesis, Massachusetts Institute of Technology, 1977.
- [10] I. Macdonald, *Symmetric Functions and Hall Polynomials*, second ed., Clarendon Press, Oxford, 1995.
- [11] P. MacMahon, *Combinatory Analysis*, Cambridge University Press, Cambridge, 1915.
- [12] A. Mendes, Building generating functions brick by brick, Ph.D. Thesis, University of California, San Diego, 2004.
- [13] V. Reiner, Signed permutation statistics, *European J. Combin.* 14 (1993) 553–567.